JOURNAL OF DIFFERENTIAL EQUATIONS, 2, 449-463 (1966).

Boundary Value Problems for Ordinary Nonlinear Second Order Systems[1]

KENNETH A. HEIMES

Duke University, Durham, North Carolina

Received January 12, 1966

1. INTRODUCTION

Using the theory of subfunctions, Fountain and Jackson construct in [*1*] a "generalized" solution of the scalar boundary value problem (BVP) $y'' = f(x, y, y')$, $y(a) = \alpha$, $y(b) = \beta$ in the sense that if a solution exists it will be this "generalized" solution by uniqueness. An investigation of the properties of this "generalized" solution enables them to impose restrictions on $f(x, y, y')$ which assure the existence of a solution to the BVP. This paper generalizes the methods and many of the results in [*1*] to second order nonlinear systems.

We adhere to the following conventions throughout. R^n denotes the space of real n-tuples with additive identity θ, norm defined by $\| y \| = \max | y_i |$ taken over all components y_i of $y \in R^n$ and with componentwise partial order ($y \geqslant \theta$ if and only if $y_i \geqslant 0$ for $i = 1,..., n$). $\Omega \in R^n$ is defined by $\Omega = (1, 1,..., 1)$. Let $T \equiv [a, b] \times R^n \times R^n$ where $[a, b]$ is a compact real interval; f defined on T into R^n is always assumed continuous. To avoid excessive subscripting in the proofs below, both x and t are used exclusively as real variables. When used together, it will be clear from the context which is considered fixed.

Consider the systems

(1) $y'' = f(x, y, y')$ and

(2) $y'' = f(x, y)$ with the boundary conditions

(3) $y(a) = \alpha$, $y(b) = \beta$ where $\alpha, \beta \in R^n$ and f is defined on T. A solution to (1), (3) [or (2), (3)] is a continuous function y on $[a, b]$ into R^n satisfying (1) [or (2)] on (a, b) and the boundary conditions (3).

---

[1] This paper is part of a doctoral thesis written under the supervision of Professor Lloyd K. Jackson at the University of Nebraska and supported by a NASA Traineeship. Final preparation was supported by the project Special Research in Numerical Analysis for the Army Research Office, Durham, Contract No. DA-31-124-AROD-13, at Duke University.

The development parallels that of [1]. In Section 2 subfunctions are defined and some of their properties are listed. A maximum principle (Theorem 1) provides a criterion for the existence of C' subfunctions. A "generalized" solution, H, is defined in Section 3 as the supremum of a certain class of subfunctions and its essential properties are determined. Restrictions on f in (1) which make H a solution to (1), (3) are then easily obtained. The existence theorems in the concluding section indicate some diversity in these restrictions.

In the sequel, reference will be made to the following conditions:

A_1 : $f(x, y, z)$ is Lipschitz in z on compact subsets of T and, for each component f_k of f, $f_k(x, u, u') \geqslant f_k(x, v, v')$ whenever $u_k' = v_k'$ and $u_k - v_k = \max_j\{u_j - v_j\} \geqslant 0$.

A_2 : For each component f_k of f, $f_k(x, u, u') > f_k(x, v, v')$ whenever $u_k' = v_k'$, and $u_k - v_k = \max_j\{u_j - v_j\} > 0$.

B_1 : There exists $K > 0$ with $\|f(x, \theta, z) - f(x, \theta, \theta)\| \leqslant K \| z \|$ for $a \leqslant x \leqslant b$, all $z \in R^n$.

B_2 : There exists $M > 0$ such that $f_k(x, \theta, z) \leqslant f_k(x, \theta, \theta)$ when $z_k \leqslant -M$ and $f_k(x, \theta, z) \geqslant f_k(x, \theta, \theta)$ when $z_k \geqslant M$ for $k = 1, 2,..., n$.

B_3 : There exists $M > 0$ such that $f_k(x, \theta, z) \geqslant f_k(x, \theta, \theta)$ when $z_k \leqslant -M$ and $f_k(x, \theta, z) \leqslant f_k(x, \theta, \theta)$ when $z_k \geqslant M$ for $k = 1, 2,..., n$.

2. Subfunctions

In this section I denotes an open, closed or half-open subinterval of $[a, b]$. $\bar{I}$ is the closure of I, I^0 the interior of I. If $\Sigma \subset R^n$ is bounded, then

$$\sup_{\sigma\in\Sigma} \sigma \equiv (\sup_{\sigma\in\Sigma} \sigma_1 ,..., \sup_{\sigma\in\Sigma} \sigma_n)$$

where σ_k is the kth component of σ. $\inf_{\sigma\in\Sigma} \sigma$ is similarly defined. When Σ is finite, replace sup by max and inf by min. For $s(x)$ defined on I into R^n,

$$s(x \pm 0) \equiv \lim_{\delta\to 0^+} s(x \pm \delta)$$

(componentwise) provided $\lim_{\delta\to 0} s_k(x \pm \delta)$ exists for each component s_k of s. Finally, let

$$\bar{D}s(x) \equiv \limsup_{\delta\to 0}\{s(x + \delta) - s(x - \delta)\}/2\delta$$

$$\underline{D}s(x) \equiv \liminf_{\delta\to 0}\{s(x + \delta) - s(x - \delta)\}/2\delta,$$

again where limits are taken componentwise.

DEFINITION 1. *A function $s(x)$ defined on I into R^n is called a subfunction on I with respect to* (1) [*or* (2)] *in case $s(x) \leqslant y(x)$ on $[x_1, x_2]$ for any subinterval $[x_1, x_2]$ of I and any solution $y(x)$ of* (1) [*or* (2)] *with $s(x_i) \leqslant y(x_i)$ $i = 1, 2$. A superfunction, $S(x)$, is defined similarly by reversing the inequalities.*

LEMMA 1. *Given positive real constants M, N and ϵ, there is a $\delta = \delta(M, N, \epsilon) > 0$ such that for any (x_1, Y_1), (x_2, Y_2) in $[a, b] \times R^n$ with $0 < x_2 - x_1 \leqslant \delta$, $\| Y_1 \| \leqslant M$, $\| Y_2 \| \leqslant M$, and $\| Y_2 - Y_1 \| \leqslant N(x_2 - x_1)$,* (1) *has at least one solution $y(x)$ on $[x_1, x_2]$ with $y(x_1) = Y_i$ for $i = 1, 2$ and $\| y(x) - \omega(x) \| \leqslant \epsilon$, $\| y'(x) - \omega'(x) \| \leqslant \epsilon$ on $[x_1, x_2]$ where*

$$\omega(x) = [(x_2 - x)Y_1 + (x - x_1)Y_2]/(x_2 - x_1).$$

In the case of Eq. (2), *the conclusion is valid without the condition*

$$\| Y_2 - Y_1 \| \leqslant N(x_2 - x_1).$$

Proof. y is a solution to (1) with $y(x_i) = Y_i$ if and only if $z = y - \omega$ is a solution to $L(z) = z$ with $z(x_i) = \theta$, where L is defined by

$$L(z)(x) = \int_{x_1}^{x_2} G(x, s) f(s, z + \omega, z' + \omega')\, ds$$

for z of class C' on $[x_1, x_2]$ into R^n and $G(x, s)$ the Green's function for the scalar equation $y'' = 0$. Let

$$K = \sup\{\| f(x, y, z) \| : a \leqslant x \leqslant b, \quad \| y \| \leqslant 2M, \quad \| z \| \leqslant 2N\}$$

and pick $\delta > 0$ so that

$$K(\delta^2 + 3\delta/2) \leqslant \min\{M, N, \epsilon\}.$$

Let H denote those C' functions $h(x)$ on $[x_1, x_2]$ into R^n satisfying $h(x_i) = \theta$, $\| h \|_0 \leqslant \min\{M, N\}$, where $\| h \|_0 = \sup\{\| h(x) \|\} + \sup\{\| h'(x) \|\}$. Then H and L satisfy the hypothesis of a fixed point theorem ([*2*], p. 119) so z (and hence y) exists. The estimates on $y - \omega$ and $y' - \omega'$ are straightforward.

The next Lemma summarizes those properties of subfunctions which can be derived directly from Lemma 1 and Definition 1. Since the proof involves componentwise adaptations of the arguments applied in Theorems 1–6 of [*1*], it is omitted.

LEMMA 2. (i) *If s is a bounded subfunction on I with respect to* (2) *then s is continuous on I^0.*

(ii) *If s is a bounded subfunction on I with respect to* (1), *then s has one-sided* *limits on* $\bar{I}$, *at most countably many discontinuities on* I, *a finite derivative almost everywhere on* I *and satisfies* $s(x) \leqslant \max\{s(x+0, s(x-0)\}$ *for* $x \in I^0$.

(iii) *If* Σ *is any collection of subfunctions on* I *bounded above at each point of* I, *then* $s(x) \equiv \sup_{\sigma \in \Sigma} \sigma(x)$ *is a subfunction on* I.

(iv) *Let* s_1 *be a subfunction on* I, s_2 *a subfunction on* $[x_1, x_2] \subset \bar{I}$, *with* $s_2(x_i) \leqslant s_1(x_i)$ *for* $i = 1, 2$. *Then*

$$s(x) \equiv \begin{cases} s_1(x) & \text{for} \quad x \notin [x_1, x_2] \\ \max\{s_1(x), s_2(x)\} & \text{for} \quad x \in [x_1, x_2] \end{cases} \quad \textit{is a subfunction on } I.$$

(v) *If* s *is a subfunction of class* C' *on* I, *then* $Ds'(x) \geqslant f(x, s(x), s'(x))$ *on* I^0.

For the inequality $\underline{D}s'(x) \geqslant f(x, s(x), s'(x))$ on I^0 to imply that a class C' function s is a subfunction on I, it is necessary to impose restrictions on f other than continuity (see [*1*], p. 1259 for the scalar case). Conditions sufficient to permit this implication are given in

THEOREM 1. *Let* f *in* (1) *satisfy* A_1 *or* A_2 *on* T. *If* u, v *are continuous on* $\bar{I}$, *of class* C' *in* I^0 *and satisfy*

(i) $\underline{D}u'(x) \geqslant f(x, u(x), u'(x)) \qquad x \in I^0$

(ii) $\bar{D}v'(x) \leqslant f(x, v(x), v'(x)) \qquad x \in I^0$

(iii) $u - v \leqslant \lambda\Omega$ *at the end points of* I *for scalar* $\lambda \geqslant 0$, *then* $u(x) - v(x) \leqslant \lambda\Omega$ *on* $\bar{I}$.

Proof. If f satisfies A_1, $f(x, u, u') \geqslant f(x, u - \lambda\Omega, u')$ for $\lambda \geqslant 0$ so the substitution $u - \lambda\Omega$ for u shows that we may assume $\lambda = 0$. This done, suppose that some component of $u - v$ has a positive interior maximum. Then there exists $x \in I^0$, an integer k and an $\epsilon > 0$ such that $4\epsilon = u_k(x) - v_k(x) \geqslant u_j(t) - v_j(t)$ for all $t \in \bar{I}$ and $j = 1, 2,..., n$. Select $[x_1, x_2] \subset I^0$ so that $u(x_i) - v(x_i) \leqslant \epsilon\Omega$ for $i = 1, 2$, where $x \in (x_1, x_2)$. Let

$$\| u(t)\| + \| u'(t)\| + \epsilon + 1 \leqslant R \text{ on } [x_1, x_2]$$

and let $K > 0$ be the Lipschitz constant for f associated with

$$\{(t, y, z) : x_1 \leqslant t \leqslant x_2, \quad \| y \| \leqslant R, \quad \| z \| \leqslant R\}.$$

Let $z(t)$ be a solution to the scalar equation $z'' = (K + 1)z'$ with $0 \leqslant z \leqslant \epsilon$, $-1 \leqslant z' < 0$ on $[x_1, x_2]$. Put $\omega(t) = u(t) - (z(t) + \epsilon)\Omega$. Then

$$\begin{aligned} \underline{D}\omega' - f(t, \omega, \omega') &\geqslant f(t, u, u') - z''\Omega - f(t, \omega, \omega') \\ &\geqslant f(t, \omega, u') - f(t, \omega, \omega') - (K + 1)z'\Omega \\ &\geqslant -K \| u' - \omega' \| \Omega - (K + 1)z'\Omega = -z'\Omega > \theta \end{aligned}$$

so that

$$\underline{D}\omega' > f(t, \omega, \omega') \text{ on } [x_1, x_2].$$

Now $\omega - v \leqslant \theta$ at x_1 and x_2 while $\omega_k(x) - v_k(x) \geqslant 2\epsilon$. Hence there is an ith component of $\omega - v$ and an $x_0 \in (x_1, x_2)$ such that

$$\omega_i(x_0) - v_i(x_0) \geqslant \sup\{\omega_j(t) - v_j(t): x_1 \leqslant t \leqslant x_2, j = 1, 2,..., n\} \geqslant 2\epsilon.$$

Then $\omega_i'(x_0) = v_i'(x_0)$ and

$$\underline{D}(\omega_i - v_i)'(x_0) > f_i(x_0, \omega, \omega') - f_i(x_0, v, v') \geqslant 0$$

by A_1, a contradiction.

When f satisfies A_2, the proof is obvious.

If f satisfies the hypothesis of Theorem 1, it is clear that solutions to (1), (3) are unique, that a function s of class C' on I into R^n is a subfunction on I if and only if $\underline{D}s'(x) \geqslant f(x, s(x), s'(x))$ on I^0 and that $s(x) - \lambda\Omega$ is a subfunction on I for scalar $\lambda \geqslant 0$ whenever s is a C' subfunction on I.

For $f(x, y, z) = A(x)z + B(x)y$ where A and B are continuous $n \times n$ matrices on $[a, b]$, condition A_1 is equivalent to having A diagonal and requiring that B have nonpositive nondiagonal entries with nonnegative row sums. The sole purpose of Theorem 2 is to show that condition A_1 is "almost" necessary in the sense that if Theorem 1 is to hold in the linear case then A must be diagonal and B must have nonpositive nondiagonal entries. Consequently, the proof of Theorem 2 is merely outlined.

We note in passing that if the right hand side of $y'' = Ay' + By$ satisfies A_1 then solutions are unique (Theorem 1) and hence always exist for any boundary values (3). (cf. [3], p. 419)

THEOREM 2. *Let A and B be continuous matrices on $[a, b]$. If, for every subinterval I of $[a, b]$, we require that every function $y(x)$ of class C^2 on I into R^n with $y'' \geqslant Ay' + By$ on I and $y \leqslant \theta$ at the endpoints of I satisfies $y \leqslant \theta$ on I, then $a_{ij}(x) = 0$ and $b_{ij}(x) \leqslant 0$ on $[a, b]$ when $i \neq j$.*

Proof. (a) Suppose that for a fixed integer $i \geqslant 1$ and fixed $x \in (a, b)$ $a_{ij}(x) \neq 0$ for at least one $j \neq i$. One can then select $\lambda > 0$ sufficiently small so that $y(t)$ defined componentwise by $y_i(t) = \lambda^2 - (t - x)^2$

$$y_j(t) = \begin{cases} \dfrac{(t - x - 2\lambda)^2}{9\lambda^2} - 1 & \text{if } \ a_{ij}(x) > 0, j \neq i \\ \dfrac{(t - x + 2\lambda)^2}{9\lambda^2} - 1 & \text{if } \ a_{ij}(x) \leqslant 0, j \neq i \end{cases}$$

satisfies $y'' \geqslant Ay' + By$ on $[x - \lambda, x + \lambda] \subset [a, b]$ with $y(x \pm \lambda) \leqslant \theta$. Since

$y(x) \nleqslant \theta$, the contradiction is clear. The conclusion holds on $[a, b]$ by continuity.

(b) Using (a) (i.e., assume A is diagonal), suppose $b_{ij}(x) > 0$ for fixed integers $i \neq j$ and fixed $x \in (a, b)$. One can select $\gamma > 0$ sufficiently large and $\lambda > 0$ sufficiently small so that $y(t)$ defined componentwise by

$$\begin{aligned} y_i(t) &= \lambda^2 - (t - x)^2 \\ y_j(t) &= (t - x)^2/\lambda^2 - \gamma \\ y_k(t) &= (t - x)^2/\lambda^2 - 1 \qquad \text{for} \quad k \neq i, j \end{aligned}$$

satisfies $y'' \geqslant Ay' + By$ on $[x - \lambda, x + \lambda] \subset [a, b]$ with $y(x \pm \lambda) \leqslant \theta$. Again $y(x) \nleqslant \theta$ and the theorem follows.

The next lemma will be useful later in bounding from above a certain class of continuous subfunctions.

LEMMA 3. *Let f in* (1) *satisfy A_1 or A_2 on T. Let S be a superfunction of class C' on $(x_1, x_2) \subset [a, b]$ with $\bar{D}S'(x) < f(x, S(x), S'(x))$ on (x_1, x_2). If s is any continuous subfunction on $[x_1, x_2]$ with $s(x_i) \leqslant S(x_i)$ for $i = 1, 2$ then $s \leqslant S$ on $[x_1, x_2]$.*

Proof. If the conclusion is false, there is a kth component of $s - S$ and an $x \in (x_1, x_2)$ such that

$$s_k(x) - S_k(x) = \sup\{s_j(t) - S_j(t) : x_1 \leqslant t \leqslant x_2, \, 1 \leqslant j \leqslant n\} = \gamma > 0.$$

Pick $\lambda > 0$ so that $[x - \lambda, x + \lambda] \subset (x_1, x_2)$. Select M and N so that $\| S(t)\| \leqslant M$, $\| S'(t)\| \leqslant N$ on $[x - \lambda, x + \lambda]$ and choose $\delta = \delta(M, N, 1) \leqslant \lambda$ as in Lemma 1. Let $y(t)$ be a solution to (1) on $I \equiv [x - \frac{1}{2}\delta, x + \frac{1}{2}\delta]$ satisfying $y(x \pm \frac{1}{2}\delta) = S(x \pm \frac{1}{2}\delta) + \gamma\Omega$. Then $y(t) < S(t) + \gamma\Omega$ in I^0 and since s is a subfunction with $s(x \pm \frac{1}{2}\delta) \leqslant y(x \pm \frac{1}{2}\delta)$ we have $s(t) < S(t) + \gamma\Omega$ in I^0, a contradiction.

3. A GENERALIZED SOLUTION

In this section we investigate the existence and properties of the supremum H of a certain class of subfunctions. When f satisfies A_1 or A_2 on T and H exists as a bounded function on $[a, b]$, H satisfies (1) almost everywhere on $[a, b]$ and is then called a "generalized" solution of (1). By carefully considering the differentiability properties of H, additional restrictions can be determined for f in (1) to insure that H is a solution to (1) on (a, b).

DEFINITION 2. *A function $\sigma[\rho]$ on $[a, b]$ into R^n is called an underfunction [overfunction] with respect to* (1), (3) *in case $\sigma(\rho)$ is a subfunction [superfunction]* *with* $\sigma(a) \leqslant \alpha$, $\sigma(b) \leqslant \beta[\rho(a) \geqslant \alpha, \rho(b) \geqslant \beta]$. *Let* U *denote the collection of all continuous underfunctions with respect to* (1), (3). *When* U *is nonvoid, define* $H(x) = \sup\{\sigma(x) : \sigma \in U\}$ *for each* $x \in [a, b]$.

THEOREM 3. *Let* f *satisfy* A_1 *or* A_2 *in* T. *If* U *is nonvoid and there exists* $M > 0$ *such that* $\sigma(x) \leqslant M\Omega$ *for all* $\sigma \in U$, $x \in [a, b]$, *then*

(i) H *is a bounded sub- and superfunction on* $[a, b]$

(ii) H *is of class* C^2 *and a solution to* (1) *on an open subset of* $[a, b]$ *whose complement has measure zero.*

(iii) $H(x) = \min\{H(x + 0), H(x - 0)\}$ for $x \in (a, b)$.

Proof. (i) Clearly H is bounded and is a subfunction by Lemma 2 (iii). Suppose H is not a superfunction on $[a, b]$. Then there is a subinterval $[x_1, x_2]$ of $[a, b]$ and a solution $y(t)$ to (1) on $[x_1, x_2]$ with $y(x_i) \leqslant H(x_i)$ while $y_k(x) - H_k(x) = \epsilon > 0$ for some $x \in (x_1, x_2)$ and some component of $y - H$. Now the jth component of H is given by $H_j(t) = \sup\{\sigma_j(t) : \sigma \in U\}$ where σ_j denotes the jth component of σ. Select functions $\lambda^1, \ldots, \lambda^n$ from U so that the jth component, λ_j^j, of λ^j satisfies $H_j(x_1) - \lambda_j^j(x_1) < \epsilon/4$. Let

$$\sigma_1(t) = \max_{1 \leqslant j \leqslant n} \{\lambda^j(t)\}.$$

Then $\sigma_1 \in U$ and $H(x_1) - \sigma_1(x_1) < (\frac{1}{4})\,\epsilon\Omega$. Similarly we obtain $\sigma_2 \in U$ such that $H(x_2) - \sigma_2(x_2) < (\frac{1}{4})\,\epsilon\Omega$. Let $\sigma(t) = \max\{\sigma_1(t), \sigma_2(t)\}$ on $[a, b]$ so $\sigma \in U$. It follows from Theorem 1 and the condition A_1 on f that $y(t) - (\epsilon/2)\Omega$ is a subfunction on $[x_1, x_2]$. Since $y(x_1) - (\epsilon/2)\Omega \leqslant H(x_1) - (\epsilon/2)\Omega < \sigma_1(x_1) \leqslant \sigma(x_1)$ and $y(x_2) - (\epsilon/2)\Omega \leqslant H(x_2) - (\epsilon/2)\Omega < \sigma_2(x_2) \leqslant \sigma(x_2)$, it follows from Lemma 2 (iv) that

$$\sigma^*(t) = \begin{cases} \sigma(t) & \text{if } \ t \notin [x_1, x_2] \\ \max\{\sigma(t), y(t) - (\epsilon/2)\Omega\} & t \in [x_1, x_2] \end{cases}$$

is a continuous underfunction on $[a, b]$. But at x,

$$\sigma_k^*(x) \geqslant y_k(x) - \epsilon/2 > y_k(x) - \epsilon = H_k(x),$$

which contradicts $\sigma^* \leqslant H$ on $[a, b]$. Hence, H is a superfunction on $[a, b]$.

(ii) H' exists almost everywhere on $[a, b]$ by Lemma 2 (ii). If $H'(x)$ is finite for $x \in (a, b)$ then Lemma 1 asserts the existence of a $\delta > 0$ and a solution y to (1) on $[x - \delta, x + \delta]$ with $y(x \pm \delta) = H(x \pm \delta)$. Since H is both a sub- and superfunction, $H(t) = y(t)$ on $[x - \delta, x + \delta]$.

(iii) The argument is similar to that in [*1*], p. 1265.

For a scalar function $g(x)$ with right- and left-hand limits $g(x+0)$ and $g(x-0)$ on (a, b), define

$$Dg(x+0) = \lim_{t\to x^+} \frac{g(t) - g(x+0)}{t - x}$$

and

$$Dg(x-0) = \lim_{t\to x^-} \frac{g(t) - g(x-0)}{t - x}.$$

As in the scalar case, it can be shown that for a bounded subfunction $s(t)$ on I into R^n $Ds(x+0) \equiv (Ds_1(x+0),..., Ds_n(x+0))$ and $Ds(x-0)$ have meaning in the extended sense ($Ds_k(x \pm 0)$ may be $\pm\infty$).

By using Lemma 1 and part (i) of Theorem 3 it is easy to show that H is a solution to (1) in an interval about $x \in (a, b)$ when $DH(x+0)$ and $DH(x-0)$ are finite. The next theorem shows that $DH(x+0)$ is finite if and only if $DH(x-0)$ is finite.

THEOREM 4. *Under the hypothesis of Theorem* 3, *let H_k be any component of H and let $x \in (a, b)$.*

(i) *If $H_k(x) = H_k(x-0) < H_k(x+0)$ then*

$$DH_k(x+0) = DH_k(x-0) = +\infty.$$

(ii) *If $H_k(x) = H_k(x+0) < H_k(x-0)$ then*

$$DH_k(x+0) = DH_k(x-0) = -\infty.$$

(iii) *If $DH_k(a+0) < +\infty$ then $H_k(a+0) = H_k(a)$ and if $DH(a+0)$ is finite then $H(a) = H(a+0) = \alpha$.*

(iv) *If $DH_k(b-0) > -\infty$ then $H_k(b-0) = H_k(b)$ and if $DH(b-0)$ is finite then $H(b) = H(b-0) = \beta$.*

If H is continuous at x, then

(v) *$DH_k(x+0) = +\infty$ if and only if $DH_k(x-0) = +\infty$.*

(vi) *$DH_k(x+0) = -\infty$ if and only if $DH_k(x-0) = -\infty$.*

Proof. Only the proofs of (i), (iii), and (v) are given. Similar arguments apply to (ii), (iv), and (vi).

On (i). Assume $DH_k(x-0) < +\infty$. Then there exist scalars $N > 0$ and $\lambda > 0$ such that $[H_k(t) - H_k(x)]/(t - x) \leqslant N$ on $[x-\lambda, x) \subset [a, b]$. Let $L(t) = N(t-x) + H_k(x)$ so $L(t) \leqslant H_k(t)$ on $[x-\lambda, x]$. Since H is bounded, there is an $M > 0$ with $2\,\|H(t)\| \leqslant M$ on $[a, b]$ and $2\,|L(t)| \leqslant M$

on $[x - \lambda, x]$. Let $\delta = \delta(M, 2N, 1) \leqslant \lambda$ be as in Lemma 1. Then there is a solution y on $[x - \delta, x]$ to (1) with

$$y(x - \delta) = (-M,..., -M, L(x - \delta), -M,..., -M)$$
$$y(x) = (-M,..., -M, L(x) + \epsilon, -M,..., -M),$$

where

$$\epsilon = \min\{(\tfrac{1}{4})[H_k(x + 0) - H_k(x)], N\delta\}$$

and $L(x - \delta), L(x) + \epsilon$ appear as the kth components of the boundary values. By construction $y(x - \delta) \leqslant H(x - \delta)$ and since y is of class C^2 on $[x - \delta, x]$, y can be extended on a small interval to the right of x. Since $H_k(x + 0) > H_k(x)$, there is an $x^* > x$ such that $y(x^*) \leqslant H(x^*)$ and y is a solution to (1) on $[x - \delta, x^*]$. Since H is a superfunction, we must have $y(t) \leqslant H(t)$ on $[x - \delta, x^*]$ while $y_k(x) = L(x) + \epsilon > H_k(x)$, a contradiction. Hence, $DH_k(x - 0) = +\infty$.

Next assume $DH_k(x + 0) < +\infty$. Then there exist scalars $N > 0$, $\lambda > 0$ such that $L(t) = N(t - x) + H_k(x + 0) \geqslant H_k(t)$ on $[x, x + \lambda] \subset [a, b]$. For M, ϵ and δ as above, there is a solution y to (1) on $[x, x + \delta]$ with

$$y(x) = (M,..., M, L(x) - \epsilon, M,..., M)$$
$$y(x + \delta) = (M,..., M, L(x + \delta), M,..., M),$$

where $L(x) - \epsilon$, $L(x + \delta)$ appear as the kth components of the boundary values. Now $y(x) \geqslant H(x)$ and $y(x + \delta) \geqslant H(x + \delta)$ so $y \geqslant H$ on $[x, x + \delta]$ since H is a subfunction. But then

$$H_k(x + 0) \leqslant y_k(x + 0) = y_k(x) = L(x) - \epsilon = H_k(x + 0) - \epsilon,$$

a contradiction.

On (iii). Suppose first that $DH_k(a + 0) < +\infty$ but that $H_k(a) < H_k(a + 0)$. (Since H is the supremum of continuous functions, $H_k(a) \leqslant H_k(a + 0)$.) Then there are scalars $N > 0, \lambda > 0$ such that $L(t) = N(t - a) + H_k(a + 0) \geqslant H_k(t)$ on $[a, a + \lambda]$. Choose $M > 0$ so that $\| H(t)\| \leqslant M$ on $[a, b]$ and $2\,| L(t)| \leqslant M$ on $[a, a + \lambda]$. Let $\delta = \delta(M, 2N, 1)$ be as in Lemma 1. Since L has slope N, there are scalars u and v such that $H_k(a) < u < H_k(a + 0) = L(a)$, $H_k(a + \delta) \leqslant L(a + \delta) < v$ with $0 < v - u < 2N\delta$ and $| u | \leqslant M$, $| v | \leqslant M$. By Lemma 1 there is a solution y to (1) on $[a, a + \delta]$ with

$$y(a) = (M,..., M, u, M,..., M)$$
$$y(a + \delta) = (M,..., M, v, M,..., M),$$

where u and v are the kth components of the boundary values. Since H is a subfunction, $H(t) \leqslant y(t)$ on $[a, a+\delta]$. But $y_k(a) < H_k(a+0)$ so $y_k(t) < H_k(t)$ for some $t \in (a, a+\delta)$, a contradiction.

Next suppose $DH(a+0)$ is finite. Then there exist scalars $N > 0$, $\lambda > 0$ such that

$$-N\Omega < \frac{H(t) - H(a)}{t-a} < N\Omega \qquad \text{on} \qquad (a, a+\lambda] \subset [a, b].$$

Let $M = N\lambda + \sup\{\| H(t)\| : a \leqslant t \leqslant b\}$ and let $\delta = \delta(M, N, 1) \leqslant \lambda$ be as in Lemma 1. Now $H(a) - N\delta\Omega < H(a+\delta)$ so there is a $\sigma \in U$ such that $H(a) - N\delta\Omega < \sigma(a+\delta) \leqslant H(a+\delta)$. Let $L(t) = N(a+\delta-t)\Omega + \sigma(a+\delta)$ on $[a, a+\delta]$ so that $L(a) > H(a)$, $L(a+\delta) = \sigma(a+\delta)$. By Lemma 1 there is a solution y to (1) on $[a, a+\delta]$ with $y(a) = \min\{\alpha, L(a)\}$ (componentwise) $y(a+\delta) = \sigma(a+\delta)$. Then $\sigma^*(t)$ defined on $[a, b]$ by

$$\sigma^*(t) = \begin{cases} \max\{y(t), \sigma(t)\} & a \leqslant t \leqslant a+\delta \\ \sigma(t) & a+\delta \leqslant t \leqslant b \end{cases}$$

is a continuous underfunction on $[a, b]$. We are forced to conclude that $H(a) = y(a) = \alpha < L(a)$.

On (v). Let $DH_k(x+0) = +\infty$ and assume that $DH_k(x-0) < +\infty$. Then there exist scalars $N > 0$, $\lambda > 0$ such that

$$\frac{H_k(t) - H_k(x)}{t-x} < N \qquad \text{on} \qquad [x-\lambda, x).$$

Hence $L(t) = N(t-x) + H_k(t)$ on $[x-\lambda, x)$. Pick $M \geqslant \| H(t)\| + |L(t)|$ on $[a, b]$ and let $\delta = \delta(M, 2N, 1) \leqslant \lambda$ be as in Lemma 1. Now H_k is continuous at x and since $DH_k(x+0) = +\infty$ there is a $\mu \in (0, \delta)$ such that $(H_k(t) - H_k(x))/t - x > 2N + 1$ and $H_k(x) < H_k(t) \leqslant H_k(x) + \delta N$ on $(x, x+\mu]$. Let $\mu^* = \delta - \mu > 0$ and let y be a solution to (1) on $[x-\mu^*, x+\mu]$ with

$$y(x-\mu^*) = (-M,\ldots, -M, L(x-\mu^*), -M,\ldots, -M)$$

$$y(x+\mu) = (-M,\ldots, -M, H_k(x+\mu), -M,\ldots, -M),$$

where $L(x-\mu^*)$ and $H_k(x+\mu)$ appear as the kth components of the boundary values. By Lemma 1, $\| y(t) - \omega(t)\| \leqslant 1$ and $\| y'(t) - \omega'(t)\| \leqslant 1$ on $[x-\mu^*, x+\mu]$, where

$$\omega(t) = [(x+\mu-t)y(x-\mu^*) + (t-x+\mu^*)y(x+\mu)]/\delta.$$

Since H is a superfunction, $y \leqslant H$ on $[x - \mu^*, x + \mu]$. In particular, $y_k(x) \leqslant H_k(x)$ so

$$2N + 1 < \frac{H_k(x + \mu) - H_k(x)}{\mu} = \frac{y_k(x + \mu) - H_k(x)}{\mu}$$

$$\leqslant \frac{y_k(x + \mu) - y_k(x)}{\mu} = y_k{}'(\gamma)$$

for some $\gamma \in (x, x + \mu)$. But $\omega_k{}'(t) < 2N$ so $y_k{}'(\gamma) - \omega_k{}'(\gamma) > 1$ which contradicts $\| y'(\gamma) - \omega'(\gamma)\| \leqslant 1$. Hence $DH_k(x - 0) = +\infty$.

The converse is similar.

4. Existence Theorems

In this concluding section we apply the preceding machinery to determine additional conditions on f which will insure that H exists and is a solution to (1), (3).

Lemma 4. *Let f in (1) satisfy A_1 or A_2 on T. If f satisfies any of B_1, B_2 or B_3, then H exists and satisfies the conclusions of Theorem* 3.

Proof. For α, β, given in (3), let $\lambda = \max\{\alpha_i, \beta_i, 0\}$, $\mu = \min\{\alpha_i, \beta_i, 0\}$. Let $R = \sup \|f(x, \theta, \theta)\|$ on $[a, b]$. From Lemma 3 it suffices to construct a continuous underfunction σ and an overfunction ρ of class C^2 on $[a, b]$ with $\rho''(x) < f(x, \rho(x), \rho'(x))$ on (a, b). If f satisfies B_1, let

$$\rho(x) = \{\lambda + r(e^{m(b-a)} - e^{m(x-a)})\}\Omega$$

and

$$\sigma(x) = \{\mu + r(e^{m(x-a)} - e^{m(b-a)})\}\Omega,$$

where $m = K + 1$, $r = R/m + 1$. If f satisfies B_2, let

$$\rho(x) = \{\lambda + A(e^{b-a} - e^{b-x})\}\Omega$$

and

$$\sigma(x) = \{\mu + A(e^{b-x} - e^{b-a})\}\Omega,$$

where $A = \max\{R, M\}$ for M given in B_2. If f satisfies B_3, let

$$\rho(x) = \{\lambda + A(e^{b-a} - e^{x-a})\}\Omega$$

and

$$\sigma(x) = \{\mu + A(e^{x-a} - e^{b-a})\}\Omega,$$

where $A = \max\{R, M\}$ for M given in B_3. In every case it is easy to verify that $\rho'' < f(x, \rho, \rho')$ and $\sigma'' > f(x, \sigma, \sigma')$ on (a, b) with $\sigma(a) \leqslant \alpha \leqslant \rho(a)$, $\sigma(b) \leqslant \beta \leqslant \rho(b)$.

THEOREM 5. *Let f in* (1) *satisfy A_1 on T. If, for each $R > 0$, there is a constant $K = K(R)$ such that $\| f(x, y, z) - f(x, y, \theta)\| \leqslant K \| z \|$ for $a \leqslant x \leqslant b$, $\| y \| \leqslant R$ and all $z \in R^n$, then* (1), (3) *has a unique solution for every choice of α, β.*

Proof. Since f satisfies B_1 on T, H exists as a bounded function and is a solution to (1) almost everywhere on $[a, b]$. Let

$$R = \sup\{\| H(x)\| : a \leqslant x \leqslant b\}, K = K(R)$$

as above and

$$N = \sup\{\| f(x, y, \theta)\| : a \leqslant x \leqslant b, \| y \| \leqslant R\}.$$

Let $x \in (a, b)$ be a point at which $H'(x)$ is finite and let $(c, d) \subset [a, b]$ denote the maximal interval about x on which H is a solution to (1). Then

$$H'(t) = H'(x) + \int_x^t f(s, H(s), H'(s))\, ds \qquad \text{for} \quad t \in (c, d).$$

Since $\| f(s, H(s), H'(s))\| \leqslant K \| H'(s)\| + N$, an application of Gronwall's inequality ([3], p. 24) shows that both $DH(d - 0)$ and $DH(c + 0)$ are finite. It follows from Theorem 4 that $c = a$, $d = b$ and H is a solution to (1), (3).

COROLLARY. *Let f in* (2) *satisfy A_1 on $[a, b] \times R^n$. Then* (2), (3) *has a unique solution for every choice of α, β.*

COROLLARY 2. *Let $g(x, y, z)$ be real valued and continuous with continuous partial derivatives with respect to y and z satisfying $(\partial g/\partial y) \leqslant 0$, $(\partial g/\partial y) + (\partial g/\partial z) \geqslant 1$ on $[a, b] \times R' \times R'$. Then the BVP*

$$y^{(4)} = g(x, y, y'') \qquad \begin{matrix} y(a) = \alpha_1, & y''(a) = \alpha_2 \\ y(b) = \beta_1, & y''(b) = \beta_2 \end{matrix} \tag{4}$$

has a unique solution.

Proof. Define f on $[a, b] \times R^2$ into R^2 by

$$f(x, y) = (y_1 - y_2, y_1 - y_2 - g(x, y_1, y_1 - y_2))$$

where $y = (y_1, y_2) \in R^2$. Since

$$f_k(x, u) - f_k(x, v) = \sum_{j=1}^{2} \int_0^1 \frac{\partial f_k}{\partial y_j}(x, su + (1 - s)v)(u_j - v_j)\, ds$$

for $u, v \in R^2$, the conditions on g insure that f satisfies A_1 on $[a, b] \times R^2$. By Corollary 1 there is a unique solution $y(x) = (y_1(x), y_2(x))$ to (2), (3) for $\alpha = (\alpha_1, \alpha_1 - \alpha_2)$, $\beta = (\beta_1, \beta_1 - \beta_2)$ and y_1 is the unique solution to (4).

DEFINITION 3. *A positive, continuous, real valued function $\phi(t)$ defined on $[0, +\infty)$ and satisfying*

$$\int_0^\infty \frac{t\, dt}{\phi(t)} = +\infty$$

is called a Nagumo function. ([4], p. 494)

THEOREM 6. *Let f in (1) satisfy A_i and B_j for some i, j and assume that for each $R > 0$ there is a Nagumo function ϕ with $\|f(x, y, z)\| \leqslant \phi(\| z \|)$ for $a \leqslant x \leqslant b$, $\| y \| \leqslant R$, all $z \in R^n$. Then (1), (3) has a unique solution for every choice of α, β.*

Proof. Since f satisfies A_i and B_j for some i and j, the generalized solution H exists, is bounded and satisfies (1) almost everywhere on $[a, b]$. If H' is bounded on any interval for which it exists, Theorem 4 asserts that H is indeed a solution to (1), (3).

Let ϕ be a Nagumo function associated with $R \equiv \sup\{\| H(x)\| : a \leqslant x \leqslant b\}$. For each component f_k of f, either A_1 or A_2 requires that $f_k(x, y, z) = f_k(x, y, \bar{z})$, where $\bar{z}$ is the vector obtained from z by replacing all but the kth component, z_k, of z with zeros. Since $\| \bar{z} \| = | z_k |$ we have $| f_k(x, y, z)| \leqslant \phi(| z_k |)$ whenever $\|f(x, y, z)\| \leqslant \phi(\| z \|)$. If H_k is any component of H and H_k' exists on $(c, d) \subset [a, b]$ then $| H_k''(x)| \leqslant \phi(| H_k'(x)|)$ on (c, d) and a standard argument ([3], p. 428) shows that H_k' is bounded on (c, d). The theorem follows.

The following "localization" of condition A_1 [or A_2], together with various restrictions on the rate of growth of f with respect to z, considerably expands the class of boundary value problems to which the above development applies. The straightforward proof that H exists and is a solution to (1), (3) is omitted.

THEOREM 7. *Let $\sigma(x)[\rho(x)]$ be a continuous underfunction [overfunction] with respect to (1), (3) with $\rho(x)$ of class C' and satisfying $\bar{D}\rho'(x) < f(x, \rho(x), \rho'(x))$ on (a, b). Assume there is a continuous function $F(x, y, z)$ satisfying A_1 [or A_2] on T with $F = f$ on*

$$T_0 \equiv [a, b] \times \{y \in R^n : \inf \sigma(x) \leqslant y \leqslant \sup \rho(x)\} \times R^n.$$

Then (1), (3) has a unique solution if any of the following conditions hold:

(i) $\|f(x, y, z)\| \leqslant \phi(\| z \|)$ on T_0 for some Nagumo function ϕ.

(ii) $\sigma(a) = \rho(a) = \alpha$ and there is an $M > 0$ such that $z_k f_k(x, y, z) \leqslant 0$ on T_0 when $| z_k | \geqslant M$ for $k = 1, 2, \ldots, n$.

(iii) σ and ρ are of class C' on $[a, b]$ with $\sigma(a) = \rho(a) = \alpha$ and $\|f(x, y, z)\| \leqslant \phi(\| z \|)$ on T_0, where ϕ is a positive, continuous real function on $[0, +\infty)$ satisfying

$$\int_\lambda^{+\infty} \frac{t\, dt}{\phi(t)} > \| \sup \rho(x) - \inf \sigma(x) \| \qquad \text{for} \quad \lambda = \max\{\| \sigma'(a)\|, \| \rho'(a)\|\}.$$

As a simple illustration, consider the 2-dimensional system

$$\begin{aligned} y_1'' &= 3y_1 - y_2^3 + g_1(x, y_1') \qquad & y_1(0) = 0, y_1(1) = 1 \\ y_2'' &= y_2 + \cos y_1 + g_2(x, y_2') \qquad & y_2(0) = 0, y_2(1) = 1. \end{aligned} \tag{5}$$

If $g_1(x, 0) \leqslant 0$, $g_1(x, 1) \geqslant 0$, $g_2(x, 0) \leqslant -1$ and $g_2(x, 1) \geqslant 0$ on $[0, 1]$, then $\sigma(x) = (0, 0)$ and $\rho(x) = (x, x)$ satisfy the hypothesis of Theorem 7. Define F on T by

$$F_1(x, y, y') = g_1(x, y_1') + \begin{cases} 3y_1 - y_2^3 & 0 \leqslant y_2 \leqslant 1 \\ 3y_1 & y_2 \leqslant 0 \\ 3y_1 - 1 & 1 \leqslant y_2 \end{cases}$$

$$F_2(x, y, y') = g_2(x, y_2') + \begin{cases} y_2 + \cos y_1 & 0 \leqslant y_1 \leqslant 1 \\ y_2 + 1 & y_1 \leqslant 0 \\ y_2 + \cos 1 & 1 \leqslant y_1 \end{cases}$$

Then F satisfies A_1 on T and if $| g_k(x, y_k')| \leqslant \phi(| y_k' |)$ for $k = 1, 2$ where $\phi(t)$ is a positive, continuous real function satisfying

$$\int_1^\infty \frac{t\, dt}{\phi(t) + 3} > 1,$$

(5) has a unique solution.

The methods in Theorems 6.1 and 6.3 of [5] can be generalized to show that if f satisfies A_1 on T and for each $R > 0$ there is an $N > 0$ such that

$$| f_k(x, y[k], z) - f_k(x, y[k], \theta)| \leqslant N | z_k |$$

for $a \leqslant x \leqslant b$, $\| y \| \leqslant R$, $z \in R^n$, $k = 1, 2, \ldots, n$, where $y[k] \equiv (y_1, \ldots, y_{k-1}, 0, y_{k+1}, \ldots, y_n)$ when $y = (y_1, \ldots, y_n)$, then $H(x)$ exists and is of class C^2 on (a, b). Moreover, H is the solution to (1), (3) when $\alpha = \beta = \theta$.

REFERENCES

1. FOUNTAIN, L., AND JACKSON, L. A generalized solution of the boundary value problem for $y'' = f(x, y, y')$. *Pacific J. Math.* **12** (1962), 1251-1272.
2. LEFSCHETZ, S. "Topics in Topology." Princeton University Press, Princeton, New Jersey, 1942.
3. HARTMAN, P. "Ordinary Differential Equations." Wiley, New York, 1964.
4. HARTMAN, P. On boundary value problems for systems of ordinary nonlinear, second order differential equations. *Trans. Amer. Math. Soc.* **96** (1960), 493-509.
5. BEBERNES, J. W. A subfunction approach to a boundary value problem for ordinary differential equations, *Pacific J. Math.* **13** (1963), 1053-1066.